

Lecture 11

Microwave instability. TMCI

January 24, 2019

Lecture outline

- Vlasov equation for the longitudinal motion, mode coupling
- Vlasov equation for the transverse motion
- Transverse mode-coupling instability (TMCI)

The Keil-Schnell-Boussard criterion gives only a crude estimate of the microwave instability threshold. A more rigorous analysis of the stability problem uses the Vlasov equation and takes into account the finite bunch length of the beam σ_z . In this lecture we will illustrate some elements of this approach. We then formulate the governing equation for the TMCI.

Vlasov equation for synchrotron oscillations

We start with the Vlasov equation (10.9),

$$\frac{\partial f}{\partial t} - c\alpha\eta \frac{\partial f}{\partial z} + K(z, t) \frac{\partial f}{\partial \eta} = 0 \quad (11.1)$$

with K given by Eq. (10.4)

$$K(z, t) = \frac{\omega_{s0}^2}{\alpha c} z - \frac{e^2}{\gamma mc} \int_z^\infty dz' n(z', t) w_\ell(z' - z) \quad (11.2)$$

Consider first the case of no wake, $w_\ell = 0$,

$$\frac{\partial f}{\partial t} - c\alpha\eta \frac{\partial f}{\partial z} + \frac{\omega_{s0}^2}{\alpha c} z \frac{\partial f}{\partial \eta} = 0 \quad (11.3)$$

Introduce cylindrical coordinates (action-angle variables) in the phase space,

$$z = r \cos \phi, \quad \frac{\alpha c}{\omega_{s0}} \eta = r \sin \phi \quad (11.4)$$

Limit of zero wake

In the cylindrical coordinates the Vlasov equation becomes³¹

$$\frac{\partial f}{\partial t} + \omega_{s0} \frac{\partial f}{\partial \phi} = 0 \quad (11.5)$$

A general solution to this equation is

$$f(r, \phi, t) = F(r, \phi - \omega_{s0}t) \quad (11.6)$$

where F is an arbitrary function periodic in ϕ with the period 2π . A steady state solution depends only on r , but if this solution is perturbed, it oscillates with harmonics of ω_{s0} . Using the periodicity, f can be also written as

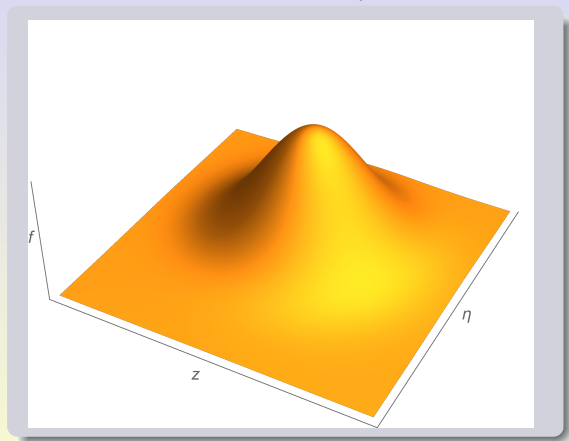
$$f(r, \phi, t) = \sum_{\ell=-\infty}^{\infty} F_{\ell}(r) e^{i\ell\phi - i\ell\omega_{s0}t} \quad (11.7)$$

with $F_{-\ell} = F_{\ell}^*$. An observer will see harmonics of the synchrotron frequency $\ell\omega_{s0}$.

³¹ Use $\frac{\partial}{\partial \phi} = \frac{\partial}{\partial z} \frac{\partial z}{\partial \phi} + \frac{\partial}{\partial \eta} \frac{\partial \eta}{\partial \phi} = -r \sin(\phi) \frac{\partial}{\partial z} + \frac{\omega_{s0}}{\alpha c} r \cos(\phi) \frac{\partial}{\partial \eta} = -\frac{\alpha c}{\omega_{s0}} \eta \frac{\partial}{\partial z} + \frac{\omega_{s0}}{\alpha c} z \frac{\partial}{\partial \eta}$

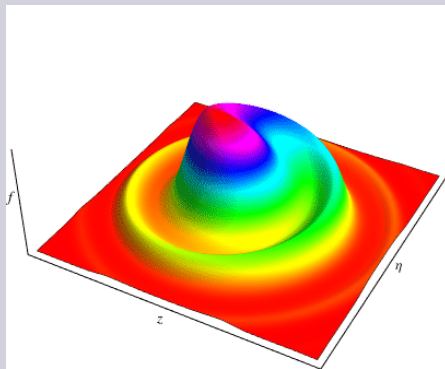
Animation of the phase space

See animation `phase_space_parabolic_potential.gif` of the phase space (we already saw this example in L9).



Effect of the frequency spread in the beam

The first effect of the longitudinal wake is that the synchrotron oscillation frequency is not constant any more, see (10.25). This introduces “mixing” of the phases. See animation `phase_space_nonlinear_potential.gif`.



This is the model where $\omega_{s0} \rightarrow \omega_{s0} + r\Delta\omega_s$ in Eq. (11.6).

Linearization and solutions of the Vlasov equation

To solve the Vlasov equation we first assume that $f(r, \phi, t) = f_0(r) + f_1(r, \phi, t)$ with $|f_1| \ll f_0$. Here f_0 is the solution of the Haïssinski equation. We linearize the equation neglecting the terms of the second and higher orders in f_1 . We then assume $f_1(r, \phi, t) \propto e^{-i\Omega t}$ and expand f_1 in a complete set of orthonormal functions $u_k(r)$, $k = 0, 1, 2, \dots$,

$$f_1(r, \phi, t) = \sum_{k,\ell} a_{k,\ell} u_k(r) e^{-i\Omega t + i\ell\phi} \quad (11.8)$$

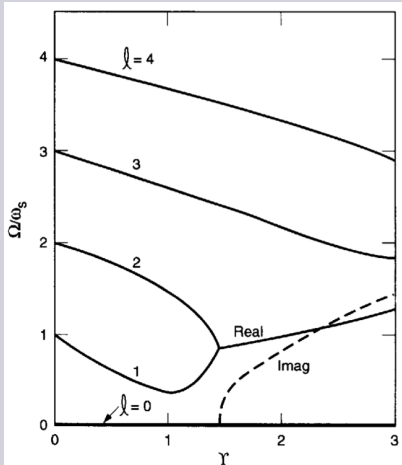
The Vlasov equation then reduces to an infinite linear system of equations for the unknowns $a_{k,\ell}$ with the matrix that involves integrals of the wake function (or the impedance). The matrix is truncated and solved for the eigenvalues — the frequency Ω is an eigenvalue of this matrix. A set of the eigenfrequencies is found as a function of the bunch charge. Typically, the system is stable if $Q < Q_{\text{thresh}}$ and unstable above the threshold.

In general, an eigenmode with an eigenfrequency Ω is a combination of all ℓ values³² — the mode *coupling*.

³²In old papers sometimes a simpler approximation was used in which it was assumed that each mode has its own ℓ and different values of ℓ do not couple. This leads to the so called Sacherer equation.

Mode coupling and the stability threshold

An example from A. Chao's book.



This example is for a diffraction model broad-band impedance ($\omega_0 = 2\pi/T_0$ with T_0 the revolution period):

$$Z_\ell(\omega) = R \left(\frac{\omega_0}{\omega} \right)^{1/2} [1 + i \operatorname{sgn}(\omega)]$$

and a model water-bag model of the beam distribution,

$$f_0(r) = \text{const}, \text{ for } r < \hat{z}$$

and $f_0 = 0$ otherwise. The parameter γ is

$$\gamma = \frac{Ne^2 \alpha R}{\gamma m \omega_{s0}^2} \left(\frac{c}{T_0 \hat{z}} \right)^{3/2}$$

Problem: use $\omega_{s0} = \alpha \sigma_{\eta} c / \sigma_z$ and compare this criterion with the Keil-Schnell Eq. (10.18).

We will now take a look at the formalism of the TMCI. In this instability it is important to take into account both the longitudinal and transverse dynamics of the bunch. In the longitudinal part, the effects of the longitudinal wake is neglected, but the short-range transverse wake is taken into account.

The synchrotron motion changes relative position of the particles in the bunch on the time scale of $\sim 1/\omega_s$. This is what makes this instability different from the BBU instability in L7.

Vlasov equation for transverse oscillations

As we know, the equations for betatron oscillations are (see Eq. (7.2))

$$\ddot{y} + \omega_{\beta}^2 y = \frac{e^2}{\gamma m} W_t \quad (11.9)$$

where W_t is the transverse wake per unit length generated by the whole beam at the location of the particle. We introduce $p = \dot{y}$ and write it as two first order equations

$$\dot{y} = p, \quad \dot{p} = -\omega_{\beta}^2 y + \frac{e^2}{\gamma m} W_t \quad (11.10)$$

If we want to describe the transverse degree of freedom only, then the distribution function is $f(y, p, t)$. The Vlasov equation is written analogous to (10.8). We first write it for the case when there is no wake,

$$\frac{\partial f}{\partial t} + \dot{y} \frac{\partial f}{\partial y} + \dot{p} \frac{\partial f}{\partial p} = \frac{\partial f}{\partial t} + p \frac{\partial f}{\partial y} - \omega_{\beta}^2 y \frac{\partial f}{\partial p} = 0 \quad (11.11)$$

Vlasov equation for transverse oscillations

Introduce the amplitude and the phase in the betatron phase space

$$\rho = \sqrt{y^2 + p^2/\omega_\beta^2} \quad (11.12)$$

and the phase variable ζ in the transverse space

$$y = \rho \cos \zeta, \quad \frac{p}{\omega_\beta} = \rho \sin \zeta \quad (11.13)$$

Considering f as a function of these variables, $f(\rho, \zeta, t)$, we find that the Vlasov equation becomes:

$$\frac{\partial f}{\partial t} - \omega_\beta \frac{\partial f}{\partial \zeta} = 0 \quad (11.14)$$

(cf. Eq (11.5))

Solving Vlasov equation without wakes

This equation can be easily solved,

$$f(\rho, \zeta, t) = F(\rho, \zeta + \omega_\beta t) \quad (11.15)$$

where F is an arbitrary function periodic in ζ with the period 2π . Using the periodicity, this can be also written as

$$f(\rho, \zeta, t) = \sum_{n=-\infty}^{\infty} F_n(\rho) e^{in\zeta + in\omega_\beta t} \quad (11.16)$$

with $F_{-n} = F_n^*$. For the average offset of the beam we have

$$\langle y \rangle = \int dp dy y f(y, p, t) = \int \rho d\rho d\zeta \rho \cos(\zeta) f(\rho, \zeta, t) \quad (11.17)$$

Note that if we compute $\langle y \rangle$ using (11.16), only terms with $n = \pm 1$ are involved, which means that $\langle y \rangle$ oscillates with the frequency ω_β .

Vlasov equation for transverse oscillations

We now take into account both the transverse and longitudinal motion. We need to consider the distribution function

$$f(y, p, z, \eta, t) \quad (11.18)$$

where again $p = \dot{y}$. It satisfies the Vlasov equation

$$\begin{aligned} \frac{\partial f}{\partial t} + \dot{z} \frac{\partial f}{\partial z} + \dot{\eta} \frac{\partial f}{\partial \eta} + \dot{y} \frac{\partial f}{\partial y} + \dot{p} \frac{\partial f}{\partial p} = & \quad (11.19) \\ \frac{\partial f}{\partial t} - c\alpha\eta \frac{\partial f}{\partial z} + \frac{\omega_{s0}^2}{\alpha c} z \frac{\partial f}{\partial \eta} + p \frac{\partial f}{\partial y} + \left(\frac{e^2}{\gamma m} W_t(z, t) - y\omega_\beta^2 \right) \frac{\partial f}{\partial p} = 0 \end{aligned}$$

In this equation we included the transverse wake, but neglected the longitudinal one. Note the arguments of W_t — here we implicitly assume an axisymmetric wake.

This is a typical starting point for analysis of transverse bunch instabilities.

Vlasov equation for the TMCI

We now change the variables from y, p, z, η to ρ, ζ, r, ϕ . The distribution function is considered as a function of these variables,

$$f(r, \phi, \rho, \zeta, t)$$

Then the Vlasov equation takes a simpler form

$$\frac{\partial f}{\partial t} + \omega_{s0} \frac{\partial f}{\partial \phi} - \omega_{\beta} \frac{\partial f}{\partial \zeta} + \frac{e^2}{\gamma m} W_t \frac{\partial f}{\partial \rho} = 0 \quad (11.20)$$

Here in the last term $W_t(z, t) \rightarrow W_t(r \cos \phi, t)$, and the derivative $\partial/\partial p$ should be expressed in terms of the derivatives $\partial/\partial \rho$ and $\partial/\partial \zeta$.

If we can neglect the wake, then

$$\frac{\partial f}{\partial t} + \omega_{s0} \frac{\partial f}{\partial \phi} - \omega_{\beta} \frac{\partial f}{\partial \zeta} = 0 \quad (11.21)$$

The general solution of this equation can be easily found

$$f(r, \phi, \rho, \zeta) = F(r, \rho, \phi - \omega_{s0}t, \zeta + \omega_{\beta}t) \quad (11.22)$$

where F is an arbitrary function of four variables periodic in ϕ and ζ with the period of 2π .

Vlasov equation for the TMCI

It can also be written as

$$f(r, \phi, \rho, \zeta, t) = \sum_{n, \ell=-\infty}^{\infty} F_{n, \ell}(r, \rho) e^{i\ell\phi + in\zeta - it(\ell\omega_{s0} - n\omega_{\beta})} \quad (11.23)$$

Similar to what we have discussed before, the average offset (at each slice z) is caused by $n = \pm 1$, so the slice centroids oscillation with the frequency $\omega_{\beta} \pm \ell\omega_{s0}$. With account of the impedance these oscillations start to couple until two of them merge resulting in an instability — the *transverse mode coupling instability*.

We now follow the derivation of M. Blaskiewicz³³. Introduce

$$g_0(z, \eta, t) = \int dy dp f(y, p, z, \eta, t) \quad (11.24)$$

and integrate the Vlasov equation (11.19) over y and p . We obtain

$$\frac{\partial g_0}{\partial t} - c\alpha\eta \frac{\partial g_0}{\partial z} + \frac{\omega_{s0}^2}{\alpha c} z \frac{\partial g_0}{\partial \eta} = \frac{\partial g_0}{\partial t} + \omega_{s0} \frac{\partial g_0}{\partial \phi} = 0 \quad (11.25)$$

³³M. Blaskiewicz. Fast head-tail instability with space charge. Phys. Rev. ST Accel. Beams, 1:044201, Aug 1998.

Vlasov equation for the TMCI

g_0 describes the longitudinal motion only. This equation is satisfied if we assume that g_0 does not depend on t and only depends on r , $g_0 = g_0(r)$. This means that we assume that longitudinally the beam remains in equilibrium. g_0 is an equilibrium longitudinal distribution in the beam (say, a Gaussian).

We then introduce two more functions

$$y_0(z, \eta, t) = \int dy dp y f \quad p_0(z, \eta, t) = \int dy dp p f. \quad (11.26)$$

Integrating Eq. (11.19) with the weights y and p we obtain

$$\begin{aligned} \frac{\partial y_0}{\partial t} + \omega_{s0} \frac{\partial y_0}{\partial \phi} - p_0 &= 0, \\ \frac{\partial p_0}{\partial t} + \omega_{s0} \frac{\partial p_0}{\partial \phi} + \omega_{\beta}^2 y_0 - \frac{e^2}{\gamma m} W_t(z, t) g_0 &= 0. \end{aligned} \quad (11.27)$$

(sometimes one replaces $\omega_{\beta}^2 \rightarrow \omega_{\beta}^2 (1 + \xi \eta)^2$, where ξ is the chromaticity) if the lattice chromaticity is taken into account).

Vlasov equation for the TMCI

We now introduce a complex variable

$$Y = y_0 + ip_0 \frac{1}{\omega_\beta} \quad (11.28)$$

If we ignore the wake and take the solution (11.22) we find that

$$Y \propto e^{-i(\omega_\beta + \ell\omega_{s0})t}.$$

The two equations (11.27) can now be combined into one equation for Y

$$\frac{\partial Y}{\partial t} + \omega_{s0} \frac{\partial Y}{\partial \phi} + i\omega_\beta Y - \frac{ie^2}{\gamma m \omega_\beta} W_t(z, t) g_0 = 0 \quad (11.29)$$

The transverse wake is proportional to the averaged over the distribution function dipole momentum of the beam and convoluted with the transverse wake w_t

$$\begin{aligned} W_t(z, t) &= \int dp dy d\eta dz' y \bar{w}_t(z' - z) f(y, p, z, \eta, t) \\ &= \int d\eta dz' y_0(z', \eta, t) \bar{w}_t(z' - z). \end{aligned} \quad (11.30)$$

The offset y_0 in this equation can be expressed as $y_0 = (Y + Y^*)/2$. We will neglect the complex conjugate term, because it is not resonant, and will use $y_0 \rightarrow Y/2$. After that Eq. (11.29) takes the form

Vlasov equation for the TMCI

Now assume $Y = Y_0(z, \eta) \exp(-i\Omega t)$. We have

$$i(\omega_\beta - \Omega)Y + \omega_{s0} \frac{\partial Y}{\partial \phi} - \frac{ie^2 N}{2\gamma m \omega_\beta} g_0 \int d\eta dz' Y(z', \eta, t) \bar{w}_t(z' - z) = 0 \quad (11.31)$$

Remember that in the last term we need to substitute $z \rightarrow r \cos \phi$.

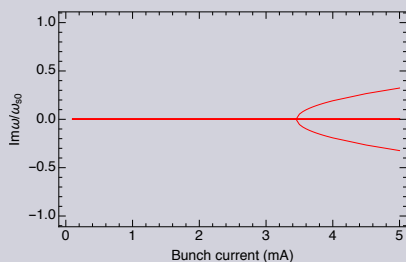
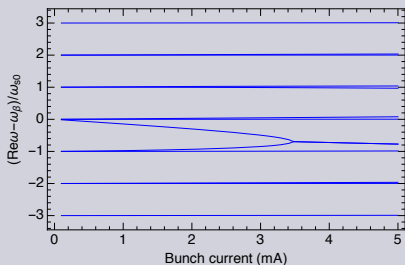
We greatly simplified our original Vlasov equation because Y_0 depends only on z and η .

There is a mode coupling effect here as well.

Sato-Chin analysis

K. Satoh and Y. Chin³⁴ developed an effective computational method for TMCI analysis assuming a Gaussian distribution of the beam.

Here is an example of stability analysis of TMCI from their paper. The resonant impedance is assumed with the resonant frequency of 1.3 GHz, $Q = 1$ and $R = 0.68 \text{ M}\Omega/\text{m}$.



³⁴ K. Satoh and Y. Chin, NIMA **207**, 207 (1983).

How to quickly estimate the threshold for TMCI?

We see that TMCI threshold corresponds to the betatron frequency shift of order of ω_{s0} . We can estimate when this happens using Eq. (11.9),

$$y\Delta\omega_\beta \sim \frac{e^2}{2\gamma m\omega_\beta} \bar{W}_t$$

For crude estimate we replace the wake by the kick factor (4.4)

$$\bar{W}_t \sim Ny\kappa_{\text{kick}} \quad (11.32)$$

which gives the following estimate for the instability threshold

$$\omega_{s0} \sim \frac{N_{\text{th}}e^2}{2\gamma m\omega_\beta} \kappa_{\text{kick}} \quad (11.33)$$

Here κ_{kick} is the kick factor per unit length.

How to quickly estimate the threshold for TMCI?

S. Krinsky³⁵ did extensive simulations for several types of impedances: a broad-band resonator, a resistive wall with normal surface impedance, and a chamber wall with extreme anomalous skin effect. He has considered: (1) the ring with a single-frequency RF system for which the equilibrium longitudinal bunch distribution is Gaussian; and (2) the ring with a third harmonic (Landau) cavity included to lengthen the bunch. His result for the threshold:

$$\frac{Ne^2\beta_y}{4\pi\gamma mc^2\nu_s} \kappa_{\text{kick}}^{\circ} \approx 0.7 \quad (11.34)$$

Here $\kappa_{\text{kick}}^{\circ}$ is the kick factor for the whole ring.

³⁵S. Krinsky, "Simulation of Transverse Instabilities in the NSLS II Storage Ring", BNL - 75019-2005-IR (2005).